ELLIPSE

Synopsis :

1. A conic section is said to be an ***ellipse*** if it's eccentricity e is less than 1.

2. The equation of an ellipse in the standard form is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

3. In the ellipse $x^2/a^2 + y^2/b^2 = 1$, $b^2 = a^2(1 - e^2)$.

4. For the ellipse $x^2/a^2 + y^2/b^2 = 1$ where a >b, there are two vertices A(a, 0), A′(–a, 0); two foci S(ae, 0), S′ (–ae, 0), two directrices x = ± a/e and two axes of which one is major axis (y = 0) and the other is minor axis (x = 0).

5. For the ellipse $x^2/a^2 + y^2/b^2=1$ where a < b, vertices are A(0, b), A′(0, –b); foci are S(0, be), S′(0, –be), directrices are y = ± b/e and the axis are x=0, y=0 (major axis and minor axis respectively).

6. A chord passing through a point P on the ellipse and perpendicular to the major axis (Principal axis) of the ellipse is called the ***double ordinate*** of the point P.

7. A chord of the ellipse passing through either of the foci of the ellipse is called a ***focal chord.***

8. A focal chord of an ellipse perpendicular to the major axis(Principal axis) of the ellipse is called ***latus rectum***. If the latus rectum meets the ellipse in L and L′ then LL′ is called ***length of the latus rectum.***

9. The length of the latus rectum of the ellipse $x^2/a^2 + y^2/b^2 = 1$ where a>b is $2b^2/a$.

10. If P is a point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ with foci S and S′ then PS + PS′ = 2a.

11. The equation of the ellipse whose major axis is parallel to x-axis and the centre at (α, β) is $\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta^2)}{b^2}$ = 1 where a > b.

12. For the ellipse $\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1$ where a >b,

    i) Centre = (α, β) ii) Eccentricity e $=\frac{\sqrt{a^2-b^2}}{a}$

    iii) Foci = (α± ae, β) iv) Vertices = (α ± a, β)

    v) Length of the latus rectum = $2b^2/a$. Equations of the latus recta are x = α ±ae.

    vi) Length of the major axis =2a. Equation of the major axis is y = β.

vii) Length of minor axis = 2b. Equation of the manor axis is x= α.

viii) Equations of the directrices are x = $\alpha \pm a/e$.

13. The equation of the ellipse whose major axis is parallel to y-axis and the centre at (α, β) is $\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1$ where a < b.

14. For the ellipse $\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1$ where a < b.

    i) Centre = (α, β) ii) Eccentricity e = $\frac{\sqrt{b^2-a^2}}{b}$

    iii) Vertices = (α, $\beta \pm b$) iv) Foci = (α , $\beta \pm be$)

    v) Length of the latus rectum = $2a^2/b$. Equation of the laturs recta are y = $\beta \pm be$.

    vi) Length of the major axis = 2b. Equation of the major axis is x =α.

    vii) Length of the minor axis = 2a. Equation of the minor axis y = β.

    viii) Equations of the directrices are y = $\beta \pm b/e$.

15. We use the following notation in this chapter.

    $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1$, $S_1 \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$, $S_{11}= S(x_1, y_1) \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$, $S_{12} \equiv \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1$.

16. Let P(x_1, y_1) be a point and $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ be an ellipse. Then

    i) P lies on the ellipse $\Leftrightarrow$ $S_{11} = 0$

    ii) P lies inside the ellipse $\Leftrightarrow$ $S_{11} < 0$

    iii) P lies outside the ellipse $\Leftrightarrow$ $S_{11} > 0$

17. The equation of the chord joining the two points A(x_1, y_1), B(x_2, y_2) on the ellipse S=0 is $S_1+S_2= S_{12}$.

18. If L = 0 is a tangent to the ellipse S = 0 at P , then we say that the line L = 0 touches the ellipse S =0 at P.

19. The equation of the tangent to the ellipse S = 0 at P(x_1, y_1) is $S_1 = 0$.

20. Let S = 0 be an ellipse and P be a point on the ellipse S = 0. The line passing through P and perpendicular to tangent of S = 0 at P is called the ***normal*** to the ellipse S = 0 at P.

21. The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$at $P(x_1, y_1)$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$.

22. The condition that the line y = mx + c may be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2 = a^2m^2 + b^2$.

23. The equation of a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may be taken as $y = mx \pm\sqrt{a^2m^2+b^2}$. The point of contact is $\left(\frac{-a^2m}{c},\frac{b^2}{c}\right)$ where $c^2 = a^2m^2 + b^2$.

24. The condition that the line lx + my + n = 0 may be a tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $a^2l^2 + b^2m^2 = n^2$.

25. Two tangents can be drawn to an ellipse from an external point.

26. If m_1, m_2 are the slopes of the tangents through $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $m_1 + m_2 = \frac{2x_1y_1}{x_1^2-a^2}, m_1m_2=\frac{y_1^2-b^2}{x_1^2-a^2}$.

27. The locus of point of intersection of perpendicular tangent to an ellipse is a circle concentric with the ellipse. This circle is called ***director circle*** of the ellipse.

28. The equation to the direction circle of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x^2 + y^2 = a^2 + b^2$.

29. The locus of the feet of the perpendiculars drawn from the foci to any tangent to the ellipse is a circle concentric with the ellipse. This circle is called ***auxiliary circle*** of the ellipse.

30. The equation of the auxiliary circle of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x^2 + y^2 = a^2$.

31. The auxiliary circle of an ellipse is the circle on the major axis of the ellipse as diameter.

32. The line joining the points of contact of the tangents to an ellipse S = 0 drawn from an external point P is called ***chord of contact*** of P with respect to the ellipse S =0.

33. The equation to the chord of contact of $P(x_1, y_1)$ with respect to the ellipse S = 0 is $S_1 = 0$.

34. The locus of the point of intersection of the tangents to the ellipse S =0 drawn at the extremities of the chord passing through a point P is a straight line L= 0, called the ***polar*** of P with respect to the ellipse S = 0. The point P is called the ***pole*** of the line L = 0 with respect to the ellipse S = 0.

35. The equation of the polar of the point $P(x_1, y_1)$ with respect to the ellipse S = 0 is $S_1 = 0$.

36. The pole of the line lx + my + n = 0 (n ≠0) with respect to the ellipse $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ is $\left(\frac{-a^2 l}{n}, \frac{-b^2 m}{n}\right)$.

37. Two points P and Q are said to be ***conjugate points*** with respect to the ellipse S = 0 if the polar of P with respect to S = 0 passes through Q.

38. The condition for the points $P(x_1, y_1)$, $Q(x_2, y_2)$ to be conjugate with respect to the ellipse S = 0 is $S_{12} = 0$.

39. Two lines $L_1 = 0$, $L_2 = 0$ are said to be ***conjugate lines*** with respect to the ellipse S =0if the pole of $L_1 = 0$ lies on $L_2 = 0$.

40. The condition for the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ to be conjugate with respect to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $a^2 l_1 l_2 + b^2 m_1 m_2 = n_1 n_2$.

41. The equation of the chord of the ellipse S = 0 having $P(x_1, y_1)$ as it's midpoint is $S_1 = S_{11}$.

42. The equation to the pair of tangents to the ellipse S =0 from $P(x_1, y_1)$ is $S_1^2 = S_{11}S$.

43. Let P(x, y) be a point on the ellipse with centre C. Let N be the foot of the perpendicular of P on the major axis. Let NP meets the auxiliary circle at P′. Then ∠NCP′ is called ***eccentric angle*** of P. The point P′ is called the corresponding point of P.

44. If θ is the eccentric angle of a point P on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and P′ is the corresponding point of P then P = (acosθ, bsinθ) , P′ = (acosθ, asinθ).

45. If P(x, y) is a point on the ellipse then x = acosθ, y = b sinθ where θ is the eccentric angle of P. These equations x = acosθ, y = bsinθ are called ***parametric equations*** of the ellipse. The point P(a cosθ, bsinθ) is simply denoted by θ.

46. The equation of the chord joining the points with eccentric angels α and β on the ellipse S = 0 is $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$.

47. The equation of the tangent at P(θ) on the ellipse S = 0 is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$.

48. The equation of the normal at P(θ) on the ellipse
S = 0 is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$.

49. Four normals can be drawn from any point to the ellipse and the sum of the eccentric angles of their feet is an odd multiple of π.

50. If the line lx + my + n =0 cuts the ellipse $x^2/a^2 + y^2/b^2 = 1$ in P and Q then the midpoint of $\overline{PQ}$ is $\left(\frac{-a^2 ln}{a^2 l^2 + b^2 m^2}, \frac{b^2 mn}{a^2 l^2 + b^2 m^2}\right)$.

51. The condition that the line $lx + my + n = 0$ to be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$.

52. A circle cuts an ellipse in four points real or imaginary. The sum of the eccentric angels of these four concyclic points on the ellipse is an even multiple of π.

53. The equation of the diameter bisecting the chords of slope m of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y = \frac{-b^2x}{a^2m}$.

54. Two diameters of an ellipse are said to be conjugate diameters. If each bisects the chords parallel to the other.

55. Two straight lines $y = m_1x$, $y = m_2x$ are conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $m_1m_2 = -b^2/a^2$